Linearized self-forces for branes

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We compute the regularized force density and renormalized action due to fields of external origin coupled to a brane of arbitrary dimension in a spacetime of any dimension. Specifically, we consider forces generated by gravitational, dilatonic and generalized antisymmetric form-fields. The force density is regularized using a recently developed gradient operator. For the case of a Nambu–Goto brane, we show that the regularization leads to a renormalization of the tension, which is seen to be the same in both approaches. We discuss the specific couplings which lead to cancellation of the self-force in this case.

I. INTRODUCTION

This article reviews and extends the use of a convenient geometric method of allowing for divergent self-interaction effects, generalizing to strings and higher branes in arbitrary spacetime dimensions the kind of regularization and renormalisation methods whose use for classical point particles in four dimensions has long been familiar. This problem was first discussed in the context of the point electron by Dirac [1], where it leads to the classical renormalization of the electron mass, and has since been seen to be a more general problem.

Our generalised method was originally developed for strings in an ordinary four dimensional background in the context of electromagnetic effects [2], with a view to its application to problems involving vortons [3], and it has since been extended to allow for the effect of linearized gravitation[4, 5, 6, 7]. For each of these interactions, it turns out that the divergent part can be dealt with just by an appropriate renormalisation of the worldsheet energy-momentum tensor, and of the relevant Lagrangian. This applies in an extensive category of classical string models [8] for which in general the string tension is less than the corresponding energy density, including the kind [9] appropriate for describing the effects of Witten's superconductivity mechanism, and also the "transonic" kind [10] describing the macroscopically averaged effects of wiggles in a Nambu–Goto string.

An important special case is that of the Nambu–Goto (NG) model itself, for which the string tension and the energy density are equal and constant, and for which it turns out that, when only gravity is involved, the kind of renormalisation developed here is not needed. In this particular case, as discussed in detail in a preceding article [5] the divergent contribution from linearized self-force simply vanishes. This result is a prototype of the cancellation theorem that has been extended to higher spacetime dimensions [11], in which the gravitational divergence does not cancel out by itself, but can be cancelled by the axion and dilaton contributions for standard values of the relevant coupling constants as obtained in the low energy limits of Superstring theories [12, 13, 14, 15].

Recently, we have developed the mathematical machinery to apply this formalism to higher-dimensional branes in a spacetime of arbitrary dimension. In [16] it was shown that the regularized gradient operator applies more generally, and this was used in the case of purely gravitational fields in [17]. The cancellation of the gravitational self-force, already known for the case of a Nambu–Goto string in four spacetime dimensions, was seen to hold for any Nambu-Goto brane of co-dimension two, whatever the spacetime dimension. It was pointed out that this could possibly provide a mechanism for self-tuning of the cosmological constant in brane-world models.

In view of the considerable interest in brane models in the context String/M-Theory both as low-energy limits and also via the fashionable concept of brane-worlds, we have generalized the calculation to include a range of coupled fields. The present discussion allows for interactions mediated by long-range linearized dilaton and form-field forces, as well as gravity. All specific cases discussed above are seen to be special cases of this generalized calculation. We compute the conditions on the couplings for the cancellation, already known between the axion and dilaton in the case of a string in four dimensions, in more general circumstances.

Before going into the technical details of the regularity problem that arises in the case of self interaction, the first few sections will be devoted to the derivation of an appropriate formula for the force density exerted by the linearized forces form-fields, the dilaton and gravity on localised brane systems of a general kind in an arbitrary number of dimensions. It is necessary to go through this carefully because, although the case of the form-field is a straightforward generalization of the Maxwell force, there are subtleties associated with the dilatonic and gravitational forces.

II. BRANE WORLDSHEET GEOMETRY

In this section we shall explain our geometrical machinery and notation. Phrasing the problem in terms of this geometric formalism allows results to be extended from specific solutions to general configurations of the brane worldsheet in an elegant way which largely avoids having to introduce new worldsheet coordinates. This section is a recapitulation of the essential geometric concepts needed for the kinematic description of the evolving worldsheet as described for cosmic strings in [18, 19]. We will be considering a p-brane (with $(p+1)$-dimensional worldsheet) in an n-dimensional "bulk" spacetime.

In this setup, the kinematics of the brane are described by the first and second fundamental tensors of the worldsheet [8, 19]. The first fundamental tensor $\eta^{\mu\nu}$ is simply the induced metric on the worldsheet in terms of the bulk coordinates. It can be written in terms of worldsheet position $\overline{x}^\mu$ and worldsheet coordinates σ_i as

$$\eta^{\mu\nu} = \gamma^{ij} \frac{\partial \overline{x}^\mu}{\partial \sigma^i} \frac{\partial \overline{x}^\nu}{\partial \sigma^j}, \qquad \gamma_{ij} = g_{\mu\nu} \frac{\partial \overline{x}^\mu}{\partial \sigma^i} \frac{\partial \overline{x}^\nu}{\partial \sigma^j}, \tag{1}$$

with γ^{ij} being the components of the worldsheet metric with respect to the worldsheet coordinates. Lowering one index gives the projection operator onto the tangent space of the brane, $\eta^\mu{}_\nu$. One can also define an orthogonal projection operator by

$$\perp^\mu{}_\nu = g^\mu{}_\nu - \eta^\mu{}_\nu. \tag{2}$$

This allows us to give a covariant expression for the radial distance, r, from the brane and the radially directed unit vector, $\nabla^\mu r$, by

$$r^2 = \perp_{\mu\nu} x^\mu x^\nu, \qquad \nabla^\mu r = r^{-1} \perp^\mu{}_\nu x^\nu. \tag{3}$$

The second fundamental tensor $K_{\mu\nu}{}^\rho$, is given by

$$K_{\mu\nu}{}^\rho = \eta^\sigma{}_\nu \overline{\nabla}_\mu \eta^\rho{}_\sigma. \tag{4}$$

where the tangentially projected differentiation operator $\overline{\nabla}_\mu$ is given by

$$\overline{\nabla}_\mu = \eta_\mu{}^\nu \nabla_\nu. \tag{5}$$

The condition of integrability of the worldsheet is the Weingarten identity, $K_{[\mu\nu]}{}^\rho = 0$, that is, that the second fundamental tensor is symmetric under interchange of its first two indices [29]. This tensor has the noteworthy property of being worldsheet orthogonal on its last index, but tangential on either of the first two indices, that is

$$K_{\mu\nu}{}^\sigma \eta_\sigma{}^\rho = 0 = \perp^\lambda{}_\mu K_{\lambda\nu}{}^\rho = 0. \tag{6}$$

The only non-vanishing trace of this second fundamental tensor is the curvature vector

$$K^\rho = K_\mu{}^{\mu\rho} = \overline{\nabla}_\nu \eta^{\nu\rho}, \tag{7}$$

which inherits worldsheet orthogonality, $\eta^\rho{}_\sigma K^\sigma = 0$. This vector K^ρ is set to zero by the dynamical equations of motion in the special case of a Nambu–Goto string or brane model, as discussed later, but not in more general models with non-trivial internal structure.

We will be discussing a model with a dilaton, so it is essential to distinguish the total Einstein metric

$$g_{\mathrm{E}\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \tag{8}$$

from the conformally related Dicke (or Jordan) metric given by $g_{\mathrm{D}\mu\nu} = e^{2\phi} g_{\mathrm{E}\mu\nu}$ where ϕ is the dilaton. Our treatment is based on linearization of all the fields, meaning that we will assume both ϕ and the metric perturbation $h_{\mu\nu}$ to be small, so that the expression

$$g_{\mathrm{D}\mu\nu} = g_{\mu\nu} + h_{\mu\nu} + 2\phi\, g_{\mu\nu}, \tag{9}$$

will be a sufficient approximation for the Dicke metric.

III.   LINEARIZED LONG-RANGE RADIATION FIELDS

The purpose of the present work is to deal with general brane models of Nambu–Goto type, for which the tension $\mathcal{T}$ of the brane is equal to the energy per unit length $\mathcal{U}$, and for which we expect long-range gravitational and electromagnetic interactions to be relevant. Most of the previous literature on string self-interaction, as well as being restricted to the Nambu–Goto case where $\mathcal{T} = \mathcal{U}$, has considered only the effects of linearized dilatonic and axionic couplings, which are physically more exotic, but technically simpler. Effects of this last kind [20] are mediated by an antisymmetric q-form gauge field, $B_{\mu_1...\mu_q}$, of the same general type that is very useful in ordinary relativistic fluid mechanics [21], with a Wess–Zumino coupling. It is possible to obtain the particular kind of fluid model appropriate for the axion case in what is known as the Zel'dovich limit: the "stiff" limit characterised by the property that perturbations propagate at the speed of light. In the general fluid case, the sound speed can be lower, but this is beyond the scope of our analysis.

Our work applies to fields satisfying the ordinary wave equation in the weak field limit, for which the background metric $g_{\mu\nu}$ is Minkowski space. These fields contribute to the n-dimensional action

$$\mathcal{I}_{\mathrm{r}} = \int \hat{L}_{\mathrm{r}} \|g\|^{1/2} \, \mathrm{d}^n x \,, \tag{10}$$

that governs the behaviour of the long range radiation, where $\hat{L}_{\mathrm{r}}$ is the kinetic contribution the Lagrangian density of the relevant fields. It is helpful to introduce a factor of $\Omega^{[n-2]}$, where $\Omega^{[n]}$ is defined as the surface area of a unit n-sphere, so as to get the usual normalization of the Newton constant in the inverse-square law. The values of $\Omega^{[n]}$ are given by

$$\Omega^{[2j+1]} = \frac{2\pi^{j+1}}{j!} \,, \qquad \Omega^{[2j]} = \frac{2^{2j}\pi^j (j-1)!}{(2j-1)!} \,. \tag{11}$$

For a given, n-dimensional background metric $g_{\mu\nu}$, the radiation action density $\hat{L}_{\mathrm{r}}$ will consist of a sum of contributions that are homogeneously quadratic functions of the gradient fields $h_{\mu\nu;\rho}$, $\phi_{;\rho}$ and $H_{\mu_1...\mu_q}$ that are respectively associated with the separate gravitational, dilatonic and q-form fields. Explicitly, the relevant gravitational, dilatonic and q-form field-strength tensors are given by

$$h_{\mu\nu;\rho} = \nabla_\rho h_{\mu\nu} \,, \qquad \phi_{;\rho} = \nabla_\rho \phi \,, \qquad H_{\mu_0\mu_1...\mu_q} = (q+1)\nabla_{[\mu_0} B_{\mu_1...\mu_q]} \,, \tag{12}$$

where ∇_ρ is the operator of covariant differentiation as specified with respect to the connection specified by the background metric $g_{\mu\nu}$. The choice of connection evidently does not matter for the definition of $\phi_{;\mu}$, nor, due to the antisymmetrisation, to that of $H_{\mu_1...\mu_q}$, but it does matter for that of the gravitational field tensor $h_{\mu\nu;\rho}$. It is to be understood that the same background metric $g_{\mu\nu}$, rather than the associated Einstein metric $g_{\mathrm{E}\mu\nu}$ or the associated Dicke metric $g_{\mathrm{D}\mu\nu}$, is used throughout for index raising and lowering. The electromagnetic field is obviously an important special case of the q-form field ($q = 1$) so we need not include it explicitly.

We now consider the form of the part of the Lagrangian $\hat{L}_{\mathrm{r}}$ governing the long-range interactions. If we use units with the speed of light c and the Dirac constant $\hbar$ set to unity, the long range radiation field contribution action to the density can be expressed in terms of constant mass parameters, M_G, M_D and $M_{[q]}$, in the form

$$\hat{L}_{\mathrm{r}} = \frac{1}{2\Omega^{[n-2]}} \left\{ \frac{M_G^{n-2}}{n-2} \mathcal{R}_{\mathrm{E}}^{(2)} - M_D^{n-2} \phi^{;\mu}\phi_{;\mu} - \frac{M_{[q]}^{n-2-2q}}{(q+1)!} H^{\mu_0\cdots\mu_q} H_{\mu_0...\mu_q} \right\} \,, \tag{13}$$

where $\mathcal{R}_{\mathrm{E}}^{(2)}$ is the residual Ricci scalar contribution, as obtained from the measure weighted Ricci scalar of the ordinary Hilbert action associated with the Einstein metric (8), by taking the expansion to quadratic order in the metric perturbation $h_{\mu\nu}$ and ignoring divergences to remove second-order derivatives. This leads to the formula

$$\mathcal{R}_{\mathrm{E}}^{(2)} = \frac{1}{2} h^{\mu\nu;\rho} \left(h_{\rho\mu;\nu} - \frac{1}{2} h_{\mu\nu;\rho} - g_{\mu\nu} h^{\rho\sigma}{}_{;\sigma} + \frac{1}{2} g_{\mu\nu} h^\sigma{}_{\sigma;\rho} \right) \,. \tag{14}$$

The constant mass scales, M_G, M_D and $M_{[q]}$, should not be confused with the masses of the corresponding bosonic particles; these bosons must have very small masses for the forces to qualify as long-range and we will assume them to be massless in our treatment. The mass scale M_G is the Planck mass of the n-dimensional bulk spacetime, a term we will not use because of possible confusion with the 4-dimensional Planck mass $M_{Pl} = \mathrm{G}^{-1/2}$, where G is Newton's

constant. The mass scale of the coupling of the dilaton, M_D, is usually supposed to be very large, at least comparable with the gravitational mass. In the usual case of ordinary spacetime ($n = 4$), solar system measurements provide severe observational limits [22] on the dimensionless Brans–Dicke parameter $\omega = 2GM_D^2 + 3/2$, which must be very large compared with unity, so that M_D itself must large compared with the Planck mass. The other mass scale, $M_{[q]}$, governs the coupling to the q-form field. In the relevant 4-dimensional application to the axion field where $q = 2$, the corresponding unrationalised pseudo-salar axion coupling constant, as used by Battye and Shellard [23, 24], is related $M_{[2]}^2 = 2\pi f_a^2$. The value of this axion coupling mass scale is usually supposed to be considerably below the Planck mass.

To obtain the field equations it is necessary to work out the variational derivatives with respect to the fields on which the action contributions depend. These variational derivatives, the vanishing of which is the condition for the field equations to be satisfied in the source-free case, are given for the dilaton contribution by

$$\frac{\delta}{\delta\phi}\left(-\phi^{;\rho}\phi_{;\rho}\right) = 2\nabla_\rho\nabla^\rho\phi\,, \tag{15}$$

and for the contribution from the q-form field by

$$\frac{\delta}{\delta B_{\mu_1...\mu_q}}\left(-H^{\rho_0...\rho_q}H_{\rho_0...\rho_q}\right) = 2(q+1)\nabla_\rho H^{\rho\mu_1...\mu_q}\,. \tag{16}$$

For the gravitational contribution, with a little more work and using the vacuum property of the background metric, one obtains the formula

$$\frac{\delta}{\delta h_{\mu\nu}}\mathcal{R}_{\mathrm{E}}^{(2)} = \frac{1}{2}\nabla_\rho\nabla^\rho h^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\nabla_\rho\nabla^\rho h^{\mu\nu} - \nabla_\rho\nabla^{(\mu}h^{\nu)\rho} + \frac{1}{2}g^{\rho(\mu}\nabla_\rho\nabla^{\nu)}h + \frac{1}{2}g^{\mu\nu}\nabla_\rho\nabla_\sigma h^{\rho\sigma}\,. \tag{17}$$

It is evident that both the action and the variational derivatives are unaffected by the gauge transformations

$$B_{\mu_1\mu_2...\mu_q} \mapsto B_{\mu_1\mu_2...\mu_q} + q\nabla_{[\mu_1}\chi_{\mu_2...\mu_q]}\,, \tag{18}$$

for an arbitrary antisymmetric covector field $\chi_{\mu_2...\mu_q}$, and subject to the background metric satisfying the Einstein vacuum field equations, it can be checked that such invariance also holds for gravitational gauge transformations of the form

$$h_{\mu\nu} \mapsto h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}\,, \tag{19}$$

for an arbitrary displacement vector field ξ^μ.

IV.   LINEARIZED INTERACTIONS

We will be considering situations where the linearized long-range fields, whose kinetics are governed by the Lagrangian (13), interact linearly with a material system described by a Lagrangian contribution provided by a master function $\hat{\Lambda}$, that depends, possibly non-linearly, on a set of internal fields representing physical quantities such as currents of various kinds, as well as on the relevant background metric. We shall consider only Lagrangians which do not depend on the derivatives of the metric.

We are considering the linearized interaction, meaning that the cross-coupling is governed by an interaction Lagrangian, $\hat{L}_{\mathrm{c}}$, of the form

$$\hat{L}_{\mathrm{c}} = \frac{1}{q!}\hat{J}^{\mu_1...\mu_q}B_{\mu_1...\mu_q} + \frac{1}{2}\hat{T}^{\mu\nu}h_{\mu\nu} + \hat{T}\phi\,, \tag{20}$$

where the coefficients $\hat{J}^{\mu_1...\mu_q}$, $\hat{T}^{\mu\nu}$, and $\hat{T}$ are functions only of the internal field quantities and the background metric that are involved in the specification of the matter Lagrangian $\hat{\Lambda}$. We shall see that $\hat{T}$ is indeed the trace of $\hat{T}^{\mu\nu}$. The total action of this underlying model takes the form

$$\mathcal{I} = \int \hat{L}\,\|g\|^{1/2}\,\mathrm{d}^n x = \int \left(\hat{\Lambda} + \hat{L}_{\mathrm{r}} + \hat{L}_{\mathrm{c}}\right)\|g\|^{1/2}\,\mathrm{d}^n x\,. \tag{21}$$

Although it was originally motivated by applications to the (most practically interesting) case of ordinary spacetime, for which $n = 4$, the results developed here are particularly suitable for applications of a more speculative kind to

cases for which n, the dimension of the background spacetime, has higher values (including the values $n = 10$ and $n = 11$ that are of particular academic interest in the context of Superstring theory and M-theory). The ensuing field equations are obtained from the requirement of local invariance with respect to the variations of the linear interaction fields $B_{\mu_1\ldots\mu_q}$, $h_{\mu\nu}$ and ϕ. There are also various internal fields, which generally enter non-linearly, involved in the specification of $\hat{\Lambda}$ and also of the source coefficients $\hat{J}^{\mu_1\ldots\mu_q}$, $\hat{T}^{\mu\nu}$, and $\hat{T}$.

The current coupling to the q-form field, $\hat{J}^{\mu_1\ldots\mu_q}$, which is a generalization of an ordinary electric current vector or the vorticity flux bivector of an axion, is restricted only by the condition that it must satisfy the flux conservation law,

$$\nabla_\mu \hat{J}^{\mu\mu_2\ldots\mu_q} = 0 \,, \tag{22}$$

in order to be invariant under local Kalb–Ramond gauge transformations of the form $B_{\mu_1\mu_2\ldots\mu_q} \mapsto B_{\mu_1\mu_2\ldots\mu_q} + q\nabla_{[\mu_1}\chi_{\mu_2\ldots\mu_q]}$ for an arbitrary covector field $\chi_{\mu_2\ldots\mu_q}$. For the special case of the electromagnetic field ($q = 1$) this is simply the current conservation condition, $\nabla_\mu \hat{J}^\mu = 0$, necessary to ensure local gauge invariance under transformations of the form $B_\mu \mapsto B_\mu + \nabla_\mu\chi$ for an arbitrary scalar χ.

Unlike the q-form field source term, the gravitational source term, $\hat{T}^{\mu\nu}$, is not something whose choice admits any latitude. In order for the theory under consideration to be considered as the linearization of a generally covariant model in which the gravitational field equations are obtained by requiring invariance with respect to variations of the total (Einstein) metric, it can be seen from the form of (8) – bearing in mind the metric dependence of the measure $\|g\|^{1/2}$ in (21) – that this gravitational source term must be given by the geometric energy-momentum tensor as defined by

$$\hat{T}^{\mu\nu} = 2\|g\|^{-1/2}\frac{\partial(\hat{\Lambda}\|g\|^{1/2})}{\partial g_{\mu\nu}} = 2\frac{\partial\hat{\Lambda}}{\partial g_{\mu\nu}} + \hat{\Lambda}g^{\mu\nu} \,. \tag{23}$$

The scalar source coefficient $\hat{T}$ might have various forms for diverse scalar coupling theories that might be conceived. However, in order for the coupling to be considered properly dilatonic it must be derived from a model in which the original specification of the matter Lagrangian was specified in terms of the Dicke metric, as given in the linearized limit by (9), from which it can be seen that the resulting linearized coupling coefficient in (20) is necessarily the trace of $T^{\mu\nu}$. In the modern context of models derived in the low energy energy limit from Superstring theory or M-theory the term dilatonic is often used for theories involving scalar couplings of a more general kind but, in most cases, such complications in the underlying theory do not modify the form of the linearized limit to which the present work is restricted.

Not much can be said about the equations of motion for the internal fields characterising the material system until the form of its Lagrangian has been specified. However, independently of such details, in a background spacetime of dimension n, the equation of motion for the q-form field, using the usual Lorentz gauge condition

$$\nabla^\mu B_{\mu\mu_2\ldots\mu_q} = 0 \,, \tag{24}$$

can be expressed in the usual [20] d'Alembertian form

$$\nabla_\sigma\nabla^\sigma B_{\mu_1\ldots\mu_q} = -\Omega^{[n-2]} M_{[q]}^{2q+2-n} \hat{J}_{\mu_1\ldots\mu_q} \,. \tag{25}$$

For the gravitational perturbation field $h_{\mu\nu}$ with a flat background metric $g_{\mu\nu}$ is flat, using the de Donder gauge condition $2\nabla^\mu h_{\mu\nu} = \nabla_\nu h^\mu_\mu$, the relevant source equation has the well known form

$$\nabla_\sigma\nabla^\sigma h_{\mu\nu} = -2(n-2)\Omega^{[n-2]} M_G^{2-n} \left(\hat{T}_{\mu\nu} - \frac{1}{n-2}\hat{T}g_{\mu\nu}\right) \,, \tag{26}$$

where n is the background spacetime dimension. It is worth remarking that, whereas, the pure gravitational radiation contribution is strictly gauge invariant, the gravitational coupling contribution would be exactly gauge invariant only if $\nabla_\mu\hat{T}^{\mu\nu} = 0$, a condition which would be satisfied only in the limit of infinitely weak coupling. This means that there will be a second order discrepancy between this gravitational source equation (26) and the equation that would be obtained by rigorous application of the formula (14). Since we are working only to linear order, this will not concern us. For the dilaton field there is no similar issue of gauge and

$$\nabla_\sigma\nabla^\sigma \phi = -\Omega^{[n-2]} M_D^{2-n} \hat{T} \,, \tag{27}$$

is the relevant field equation.

V. DISTRIBUTIONAL SOURCES

There are problems inherent with modelling branes as distributional sources, where the source densities $\hat{J}^{\mu_2\cdots\mu_q}$ and $\hat{T}^{\mu\nu}$ are Dirac δ-functions vanishing outside the worldsheet. In the case where the codimension is greater than one this will give rise to an ultra-violet (UV) divergence, as is most familiar in the case $p = 0$ of a point particle. When one considers the full, non-linear Einstein equations, there are further problems with distributional sources [25] which we will not discuss here since we are only considering linearized gravity.

In the case of a general p-brane, where the brane worldsheet has the locus $x^\mu = \overline{x}^\mu\{\sigma\}$ in terms of intrinsic coordinates σ^i ($i = 0, 1, ..., p$), we can write the distributional fields $\hat{J}^{\mu_1\cdots\mu_q}$ and $\hat{T}^{\mu\nu}$ as

$$\hat{j}^{\mu_1\cdots\mu_q} = \|g\|^{-1/2} \int \overline{J}^{\mu_1\cdots\mu_q}\, \delta^{[n]}[x - \overline{x}\{\sigma\}]\, \|\gamma\|^{1/2}\, \mathrm{d}^{p+1}\sigma\,, \tag{28}$$

$$\hat{T}^{\mu\nu} = \|g\|^{-1/2} \int \overline{T}^{\mu\nu}\, \delta^{[n]}[x - \overline{x}\{\sigma\}]\, \|\gamma\|^{1/2}\, \mathrm{d}^{p+1}\sigma\,, \tag{29}$$

where $\|\gamma\|$ is the determinant of the induced metric. The field $\overline{J}^{\mu_1\cdots\mu_q}$ is the generalised surface current of the q-form field and is a regular vector field on the brane worldsheet, but undefined off it. The flux conservation law (22) will also apply to this surface current. We define $\overline{T}^{\mu\nu}$ similarly. The same is true of the Lagrangian, which can be written

$$\hat{L} = \|g\|^{-1/2} \int \overline{L}\, \delta^{\mathrm{n}}[x - \overline{x}\{\sigma\}]\, \|\gamma\|^{1/2}\, \mathrm{d}^{p+1}\sigma\,, \tag{30}$$

and similarly for the component contributions $\overline{\Lambda}$ and $\overline{L}_{\mathrm{c}}$. The master function, $\overline{\Lambda}$, will be the intrinsic worldsheet Lagrangian, which is a function just of the relevant internal fields, such as currents, on the string, and of its induced metric, while the cross coupling contribution, $\overline{L}_{\mathrm{c}}$, will be given in terms of the worldsheet confined fields $\overline{J}^{\mu_1\cdots\mu_q}$ and $\overline{T}^{\mu\nu}$ by

$$\overline{L}_{\mathrm{c}} = \frac{1}{q!}\overline{J}^{\mu_1\cdots\mu_q} B_{\mu_1\cdots\mu_q} + \frac{1}{2}\overline{T}^{\mu\nu} h_{\mu\nu} + \overline{T}\phi\,. \tag{31}$$

The action can then be expressed, without any distributional terms, as a simple $(p + 1)$-surface integral

$$\mathcal{I} = \int \overline{L}\, \|\gamma\|^{1/2}\, \mathrm{d}^{p+1}\sigma\,. \tag{32}$$

The surface energy-momentum tensor $\overline{T}^{\mu\nu}$, which appears in $\overline{L}_{\mathrm{c}}$, can be obtained directly from the worldsheet master function, $\overline{\Lambda}$, without the use of distributions, by variation with respect to the metric, giving the formula

$$\overline{T}^{\mu\nu} = 2\|\gamma\|^{-1/2}\frac{\partial(\overline{\Lambda}\|\gamma\|^{1/2})}{\partial g_{\mu\nu}}\,. \tag{33}$$

To vary the part of the action coming from $\overline{L}_{\mathrm{c}}$, we need the variational derivative of $\overline{T}^{\mu\nu}$ with respect to the metric. This motivates us to define what we call the *hyper-Cauchy tensor* (a relativistic generalisation of the Cauchy elasticity tensor of classical mechanics) which is defined by

$$\overline{C}^{\mu\nu\rho\sigma} = \|\gamma\|^{-1/2}\frac{\delta}{\delta g_{\mu\nu}}\Big(\overline{T}^{\rho\sigma}\|\gamma\|^{1/2}\Big) = 2\|\gamma\|^{-1/2}\frac{\delta}{\delta g_{\mu\nu}}\frac{\delta}{\delta g_{\rho\sigma}}\Big(\overline{\Lambda}\|\gamma\|^{1/2}\Big) = \overline{C}^{\rho\sigma\mu\nu}\,, \tag{34}$$

and is manifestly symmetric under the interchange of the first pair of indices with the second pair. We can rewrite the expressions (33) and (34) in the more practical forms

$$\overline{T}^{\mu\nu} = 2\frac{\partial\overline{\Lambda}}{\partial g_{\mu\nu}} + \overline{\Lambda}\eta^{\mu\nu}\,, \qquad \overline{C}^{\mu\nu\rho\sigma} = \frac{\delta\overline{T}^{\rho\sigma}}{\delta g_{\mu\nu}} + \frac{1}{2}\overline{T}^{\rho\sigma}\eta^{\mu\nu}\,, \tag{35}$$

and both the energy-momentum tensor and the hyper-Cauchy tensor are tangent to the brane, that is,

$$\perp^\lambda{}_\mu\overline{T}^{\mu\nu} = \perp^\lambda{}_\mu\overline{C}^{\mu\nu\rho\sigma} = 0\,. \tag{36}$$

The generalized current $\overline{J}^{\mu_1\cdots\mu_q}$ is a conserved flux, so its variation with respect to the metric is a total derivative, which will vanish in the action integral.

If the corresponding radiation fields $B_{\mu_1...\mu_q}$, $h_{\mu\nu}$ and ϕ are considered to be regular background fields due to external sources, the treatment of such a system will be straightforward, but it is evident that this will not be the case for the radiation fields produced by the brane itself, since they will be singular at the brane just where their evaluation is needed.

The introduction of these fields enables the maximally symmetric (static, asymptotically vanishing) solution of the simple wave equation (27) for a uniform p-brane supported distribution of the form (29) to be expressed in the form

$$\phi^{;\rho} = -\frac{\Omega^{[n-2]}}{\Omega^{[n-2-p]}} M_D^{2-n} \overline{T} \perp^\rho_\sigma \frac{x^\sigma}{r^{n-1-p}} \,, \tag{37}$$

in which the angle factor $\Omega^{[n-2]}/\Omega^{[n-2-p]}$ evidently reduces to unity in the case of a point particle ($p = 0$). The analogous expressions for the form and gravitational fields are given by

$$B^{\mu_1\cdots\mu_q;\rho} = -\frac{\Omega^{[n-2]}}{\Omega^{[n-2-p]}} M_{[q]}^{2q+2-n} \overline{J}^{\mu_1\cdots\mu_q} \perp^\rho_\sigma \frac{x^\sigma}{r^{n-1-p}} \,, \tag{38}$$

and

$$h^{\mu\nu;\rho} = -2(n-2)\frac{\Omega^{[n-2]}}{\Omega^{[n-2-p]}} M_G^{2-n} \left(\overline{T}^{\mu\nu} - \frac{1}{n-2}\overline{T}g^{\mu\nu}\right)\perp^\rho_\sigma \frac{x^\sigma}{r^{n-1-p}} \,. \tag{39}$$

VI.   THE FORCE DENSITY FORMULAE

To derive the equations of motion from a variation principle, we must consider perturbative displacements with respect to the background characterised by the metric $g_{\mu\nu}$ and the linearly coupled $h_{\mu\nu}$, ϕ and $H_{\mu_1\cdots\mu_q}$ fields. We find it most convenient to describe the effect of displacements using a Lagrangian treatment where the background coordinates x^μ are considered to be dragged along by the displacement, so that the relevant field variations are given just by the corresponding Lie derivatives with respect to the vector field ξ^μ describing the displacement under consideration. This leads to the formulae

$$\delta B_{\mu_1\mu_2...\mu_q} = \xi^\sigma\nabla_\sigma B_{\mu_1\mu_2...\mu_q} + qB_{\sigma[\mu_2...\mu_q}\nabla_{\mu_1]}\xi^\sigma \,, \tag{40}$$
$$\delta h_{\mu\nu} = \xi^\sigma\nabla_\sigma h_{\mu\nu} + 2h_{\sigma(\mu}\nabla_{\nu)}\xi^\sigma \,, \tag{41}$$
$$\delta\phi = \xi^\sigma\nabla_\sigma\phi \,, \tag{42}$$

for the q-form, gravitational and dilatonic fields respectively, while finally for the background metric itself one has the well known formula

$$\delta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} \,. \tag{43}$$

There are, of course, the internal fields on which the master function Λ depends and these must also be perturbed in a full variational analysis. However, if the internal field equations are satisfied, these perturbations will have no effect on the action integral $\mathcal{I}$ so, for the purpose of evaluating the variation $\delta\mathcal{I}$, there will be no loss of generality in assuming that these fields are unperturbed. The worldsheet flux conservation law (22) tells us [18] that $\overline{J}^{\mu_1\cdots\mu_q}$ is related by Hodge duality to the exterior derivatives of corresponding worldsheet $(p - q + 1)$-forms. In the usual cases these differential forms will be included among (or depend only on) the relevant internal fields whose variation we can legitimately ignore for the purpose of evaluating $\delta\mathcal{I}$, so the variation of these $(p - q + 1)$-forms can also be taken to be zero. This means that the variations of the corresponding surface density will also vanish, that is,

$$\delta\left(\|\gamma\|^{1/2}\overline{J}^{\mu_1\cdots\mu_q}\right) = 0 \,. \tag{44}$$

It follows that the contribution from the q-form field to the variation of (31) will be given by

$$\delta\left(\|\gamma\|^{1/2}\overline{J}^{\mu_1\cdots\mu_q}B_{\mu_1...\mu_q}\right) = \|\gamma\|^{1/2}\overline{J}^{\mu_1\cdots\mu_q}\,\delta B_{\mu_1...\mu_q} \,. \tag{45}$$

The variation of the background metric does not contribute to the variation of the q-form terms in (31), but it is of paramount importance for the evaluation of the corresponding contribution from the gravitational and dilatonic

coupling terms. It can be seen from (35) that the gravitational contribution to the variation of the integrand in (32) will be given by the – until recently [4] not so well known – expression

$$\delta\left(\frac{1}{2}\|\gamma\|^{1/2}\,\overline{T}^{\mu\nu}h_{\mu\nu}\right) = \frac{1}{2}\|\gamma\|^{1/2}\left(\overline{T}^{\mu\nu}\delta h_{\mu\nu} + \overline{C}^{\mu\nu\rho\sigma}h_{\rho\sigma}\delta g_{\mu\nu}\right),$$
(46)

while, despite its deceptively simple scalar nature, the dilatonic coupling gives rise to a corresponding contribution that works out to be given by the – even less well known – expression

$$\delta\left(\|\gamma\|^{1/2}\,\overline{T}\phi\right) = \|\gamma\|^{1/2}\left(\overline{T}\delta\phi + (\overline{T}^{\mu\nu} + \overline{C}^{\mu\nu})\phi\,\delta g_{\mu\nu}\right),$$
(47)

using the notation $\overline{C}^{\mu\nu} = \overline{C}^{\mu\nu\rho}{}_{\rho}$.

When the internal field equations are satisfied, the variation of the background metric will of course provide the only contribution from the term involving the master function $\overline{\Lambda}$ in the Lagrangian. As can be seen from (33), this last contribution will simply be given by an expression of the familiar form

$$\delta\left(\|\gamma\|^{1/2}\Lambda\right) = \frac{1}{2}\|\gamma\|^{1/2}\overline{T}^{\mu\nu}\delta g_{\mu\nu}.$$
(48)

To evaluate the integrated effect of these contributions (45–48) we substitute the relevant Lie derivative formulae (40–43) and write the terms involving derivatives of the displacement fields as total divergences. Using the current conservation law (22) one finds

$$\frac{1}{q!}\overline{J}^{\mu_1\cdots\mu_q}\delta\,B_{\mu_1\ldots\mu_q} = \frac{1}{q!}\xi^{\mu}H_{\mu\mu_1\ldots\mu_q}\overline{J}^{\mu_1\cdots\mu_q} + \frac{1}{(q-1)!}\overline{\nabla}_{\mu}\left(\xi^{\nu}B_{\nu\mu_2\ldots\mu_q}\overline{J}^{\mu\mu_2\cdots\mu_q}\right),$$
(49)

from (45). For the first term in (46)

$$\frac{1}{2}\overline{T}^{\mu\nu}\delta h_{\mu\nu} = \xi^{\mu}\left[\frac{1}{2}\overline{T}^{\nu\rho}\nabla_{\mu}h_{\nu\rho} - \overline{\nabla}_{\nu}\left(\overline{T}^{\nu\rho}h_{\mu\rho}\right)\right] + \overline{\nabla}_{\mu}\left(\xi^{\nu}\overline{T}^{\mu\rho}h_{\nu\rho}\right),$$
(50)

and for the second term in (46)

$$\frac{1}{2}\overline{C}^{\mu\nu\rho\sigma}h_{\rho\sigma}\delta g_{\mu\nu} = -\xi^{\mu}\overline{\nabla}_{\nu}\left(\overline{C}^{\nu}{}_{\mu}{}^{\rho\sigma}h_{\rho\sigma}\right) + \overline{\nabla}_{\mu}\left(\xi^{\nu}\overline{C}^{\mu}{}_{\nu}{}^{\rho\sigma}h_{\rho\sigma}\right).$$
(51)

The corresponding expression for the second term in (47) is

$$(\overline{T}^{\mu\nu} + \overline{C}^{\mu\nu})\phi\,\delta g_{\mu\nu} = -2\xi^{\mu}\overline{\nabla}_{\nu}\left[(\overline{T}^{\nu}{}_{\mu} + \overline{C}^{\nu}{}_{\mu})\phi\right] + 2\overline{\nabla}_{\mu}\left[\xi^{\nu}(\overline{T}^{\mu}{}_{\nu} + \overline{C}^{\mu}{}_{\nu})\phi\right],$$
(52)

while for the first term in (47) one trivially obtains

$$\overline{T}\delta\phi = \xi^{\mu}\overline{T}\nabla_{\mu}\phi.$$
(53)

Finally, for the intrinsic contribution given by (48) one obtains an expression of the familiar form

$$\frac{1}{2}\overline{T}^{\mu\nu}\delta g_{\mu\nu} = -\xi^{\mu}\overline{\nabla}_{\nu}\overline{T}^{\nu}{}_{\mu} + \overline{\nabla}_{\mu}\left(\xi^{\nu}\overline{T}^{\mu}{}_{\nu}\right).$$
(54)

To derive the force density formulae, we vary the action integral (32) with respect to ξ_{μ}, the displacement. We wish apply Green's theorem to remove the divergence terms, so we must require that the displacement be confined to a finite region. The result is an expression of the form

$$\delta\mathcal{I} = \int \xi^{\mu}\left(\overline{f}_{\mu} - \overline{\nabla}_{\nu}\overline{T}^{\nu}{}_{\mu}\right)\|\gamma\|^{1/2}\,\mathrm{d}^{p+1}\sigma,$$
(55)

so that applying the variation principle gives the equation of motion

$$\overline{\nabla}_{\nu}\overline{T}^{\mu\nu} = \overline{f}^{\mu}$$
(56)

in which the vector $\overline{f}^{\mu}$ represents the total force density exerted by the various radiation fields involved. This force density can immediately be read out in the form

$$\overline{f}^{\mu} = \overline{f}_{[q]}{}^{\mu} + \overline{f}_{\mathrm{G}}{}^{\mu} + \overline{f}_{\mathrm{D}}{}^{\mu},$$
(57)

in which the contributions from the various fields involved are as follows. The q-form contribution can be seen from (49) to be given by

$$\overline{f}_{[q]}{}^{\mu} = \frac{1}{q!} H^{\mu}{}_{\mu_1 \ldots \mu_q} \overline{J}^{\mu_1 \ldots \mu_q} \,, \tag{58}$$

which, in the case $q = 1$, is the Lorentz force of electromagnetism. The gravitational contribution can be seen from (50) and (51) to be given by

$$\overline{f}_{\mathrm{G}}{}^{\mu} = \frac{1}{2} \overline{T}^{\nu\sigma} \nabla^{\mu} h_{\nu\sigma} - \overline{\nabla}_{\nu} \left(\overline{T}^{\nu\sigma} h_{\sigma}{}^{\mu} + \overline{C}^{\mu\nu\rho\sigma} h_{\rho\sigma} \right) \,. \tag{59}$$

Finally, the dilatonic contribution can be seen from (52) and (53) to be given by the expression

$$\overline{f}_{\mathrm{D}}{}^{\mu} = \overline{T} \nabla^{\mu} \phi - 2 \overline{\nabla}_{\nu} \left[(\overline{T}^{\mu\nu} + \overline{C}^{\mu\nu}) \phi \right] \,. \tag{60}$$

It has been shown [4] that some of the early work on cosmic strings [14] is flawed due to omission of some of the terms and, consequently, the related physical effects.

If the system of equations governing the dynamics of the internal fields in the brane worldsheet involves Q independent degrees of freedom, the complete set of dynamical equations governing the evolution of the brane will involve a total of $Q + p + 1$ degrees of freedom, including those needed to determine the geometrical evolution of the $p + 1$ dimensional supporting worldsheet. Since it involves n components, the force law (56) will, by itself, provide a complete system of equations of motion for the system if $Q < n - p$. There will even be some redundancy, in the sense that the equations of the system (56) will not all be mutually independent, in cases for which $Q < n - p - 1$. In the case of a brane of the simple Nambu–Goto type for which $Q = 0$ because there are no internal fields, this is particularly so.

VII. ALLOWANCE FOR REGULARIZED SELF INTERACTION

Before proceeding further, it is convenient to decompose the various linear perturbation fields under consideration in the form

$$B_{\mu_1 \ldots \mu_q} = \widetilde{B}_{\mu_1 \ldots \mu_q} + \widehat{B}_{\mu_1 \ldots \mu_q} \,, \qquad h_{\mu\nu} = \widetilde{h}_{\mu\nu} + \widehat{h}_{\mu\nu} \,, \qquad \phi = \widetilde{\phi} + \widehat{\phi} \,, \tag{61}$$

using a tilde for the locally source free contributions $\widetilde{B}_{\mu_1 \ldots \mu_q}$, $\widetilde{h}_{\mu\nu}$ and $\widetilde{\phi}$, respectively attributable to incident q-form, gravitational and dilatonic radiation, and a hat for the contributions given by the retarded Green function solutions of the relevant source equations (25), (26) and (27). This will give rise to corresponding decompositions

$$\bar{f}_{[q]}^{\mu} = \widetilde{f}_{[q]}^{\mu} + \widehat{f}_{[q]}^{\mu} \,, \qquad \bar{f}_{\mathrm{G}}^{\mu} = \widetilde{f}_{\mathrm{G}}^{\mu} + \widehat{f}_{\mathrm{G}}^{\mu} \qquad \bar{f}_{\mathrm{D}}^{\mu} = \widetilde{f}_{\mathrm{D}}^{\mu} + \widehat{f}_{\mathrm{D}}^{\mu} \,, \tag{62}$$

for the associated force densities as specified by the general formulae (58), (59), (60).

In many contexts the coupling is so weak that the self-force contributions $\widehat{f}_{[q]}^{\mu}$, $\widehat{f}_{\mathrm{G}}^{\mu}$ and $\widehat{f}_{\mathrm{D}}^{\mu}$ can be neglected. However, in cases for which one needs to take account of the self induced contributions $\widehat{B}_{\mu_1 \ldots \mu_q}$, $\widehat{h}_{\mu\nu}$ and $\widehat{\phi}$, one runs into difficulties arising from the fact that the field equations (25), (26) and (27) will be sourced by the distributional fields, $\hat{J}^{\mu_1 \ldots \mu_q}$, $\hat{T}^{\mu\nu}$ and $\hat{T}$, rather than the regular worldsheet supported fields $\overline{J}^{\mu_1 \ldots \mu_q}$, and $\overline{T}^{\mu\nu}$. For sources such as these, the resulting field contributions will diverge in the thin worldsheet limit if $n - p > 1$.

To regularize this, we observe that the infinitely thin worldsheet is an approximation of the physical object, which has finite thickness, ϵ, giving an ultraviolet (UV) cut-off scale. This will be sufficient to regularize point particles and extended objects of codimension greater than two, that is, $n - p > 3$, in which cases the divergence will be of power law type. However, in the logarithmically divergent case of a hyperstring of codimension 2, it will also be necessary to introduce a long-range infrared (IR) cut off length scale, Δ, that might represent the macroscopic mean distance between neighbouring hyperstrings or the compactification radius of extra-dimension in a brane-world model.

We can deal with all these cases, leaving aside only the hypersurface case of codimension one, by introducing a regularization factor [16, 17] of the form

$$\mathrm{F}_{\{\Delta, \epsilon\}} = \frac{\Omega^{[n-2]} \Omega^{[p]}}{\Omega^{[n-1]}} \int_{\epsilon}^{\Delta} x^{p-n+2} \, \mathrm{d}x \,, \tag{63}$$

which should be accurate upto a factor of $\mathcal{O}(1)$. This will be proportional to ϵ^{p-n+3} (assuming Δ to be large) when $p + 3 < n$ and $\log(\Delta/\epsilon)$ when $p + 3 = n$.

It can then be seen from (25) that the regularized q-form self field contribution will be given by

$$\widehat{B}_{\mu_1...\mu_q} = \frac{1}{n-2} \mathrm{F}_{\{\Delta,\epsilon\}} M_{[q]}^{2q+2-n} \overline{J}_{\mu_1...\mu_q}. \tag{64}$$

Similarly, from (26) the corresponding expression for the regularized gravitational self-field, $\widehat{h}_{\mu\nu}$, will be

$$\widehat{h}_{\mu\nu} = 2\,\mathrm{F}_{\{\Delta,\epsilon\}} M_G^{2-n} \left(\overline{T}_{\mu\nu} - \frac{1}{n-2} \overline{T}_\sigma{}^\sigma g_{\mu\nu} \right), \tag{65}$$

while finally by (27) we find that

$$\widehat{\phi} = \frac{1}{n-2} \mathrm{F}_{\{\Delta,\epsilon\}} M_D^{2-n} \overline{T}, \tag{66}$$

for the regularized dilatonic self-field. For most purposes it will be adequate to use the same regularization factor $\mathrm{F}_{\{\Delta,\epsilon\}}$ for all the different fields. This is especially true in the case the hyperstring case, where the dependence on the cut-off is only logarithmic.

In order to obtain correspondingly regularized self-force contributions $\widehat{f}_{[q]}^{\,\mu}$, $\widehat{f}_G^{\,\mu}$ and $\widehat{f}_D^{\,\mu}$ from the formulae (58), (59) and (60), we need to know not only the regularized values of the self-fields $\widehat{B}_{\mu_1...\mu_q}$, $\widehat{h}_{\mu\nu}$ and $\widehat{\phi}$, but also the regularized values of their gradients. There is no difficulty for the terms involving just the tangentially projected gradient operator $\overline{\nabla}_\nu$ but there are also contributions from the unprojected gradient operator ∇_ν which is meaningful only when acting fields whose support extends off the worldsheet.

Fortunately, this problem has a very simple general solution [16], of which particular applications in particular gauges are implicit in much previous work [14, 23, 24, 26] and which was first formulated explicitly in the specific context of the electromagnetic force in the string case [2]. One finds, by examining the string worldsheet limit behaviour of derivatives of the relevant Green function, that the appropriate regularization of the gradients on the string worldsheet is obtained simply by replacing the ill-defined ∇_ν by the corresponding regularized gradient operator given in terms of the worldsheet curvature vector K^μ by the formula

$$\widehat{\nabla}_\nu = \overline{\nabla}_\nu + \frac{1}{2} K_\nu. \tag{67}$$

Applying (67) to the q-form force contribution in (58) one finds that it can be formulated as a worldsheet divergence and written in the form

$$\widehat{f}_{[q]}^{\,\mu} = -\overline{\nabla}_\nu \widehat{T}_{[q]}^{\,\mu\nu}, \tag{68}$$

in which

$$\widehat{T}_{[q]}^{\,\mu\nu} = \frac{1}{(q-1)!} \widehat{B}^\mu{}_{\rho_2...\rho_q} \overline{J}^{\nu\rho_2...\rho_q} - \frac{1}{2q!} \widehat{B}_{\rho_1..\rho_q} \overline{J}^{\rho_1...\rho_q} \eta^{\mu\nu}. \tag{69}$$

When one applies same procedure to the gravitational self-force contribution in (59) one finds [6] that it too can be formulated as a worldsheet divergence in the analogous form

$$\widehat{f}_G^{\,\mu} = -\overline{\nabla}_\nu \widehat{T}_G{}^{\mu\nu}, \tag{70}$$

in which the relevant energy-momentum contribution from the gravitational self-interaction works out to be given by the expression

$$\widehat{T}_G{}^{\mu\nu} = \widehat{h}_\sigma{}^\mu \overline{T}^{\nu\sigma} - \frac{1}{4} \widehat{h}_{\rho\sigma} \overline{T}^{\rho\sigma} \eta^{\mu\nu} + \widehat{h}_{\rho\sigma} \overline{\mathcal{C}}^{\rho\sigma\mu\nu}. \tag{71}$$

Similarly for the dilatonic contribution in (60) one obtains

$$\widehat{f}_D^{\,\mu} = -\overline{\nabla}_\nu \widehat{T}_D{}^{\mu\nu}, \tag{72}$$

with

$$\widehat{T}_D{}^{\mu\nu} = 2\widehat{\phi} \left(\overline{T}^{\mu\nu} - \frac{1}{4} \overline{T} \eta^{\mu\nu} + \overline{\mathcal{C}}^{\mu\nu} \right). \tag{73}$$

The remarkable fact that such a formulation exists is what makes it possible to describe the the result of this regularization as a "renormalisation": the possibility of expressing the self-force contributions as divergences implies that that they can be absorbed into the left hand side of the basic force balance equation by a renormalisation whereby the original "bare" energy-momentum tensor $\overline{T}^{\mu\nu}$ undergoes a replacement $\overline{T}^{\mu\nu} \mapsto \widetilde{T}^{\mu\nu} = \overline{T}^{\mu\nu} + \widehat{T}^{\mu\nu}$ with

$$\widehat{T}^{\mu\nu} = \widehat{T}_{[q]}{}^{\mu\nu} + \widehat{T}_{\mathrm{G}}{}^{\mu\nu} + \widehat{T}_{\mathrm{D}}{}^{\mu\nu} = \widehat{h}_\sigma{}^\mu \overline{T}^{\nu\sigma} - \frac{1}{4}\widehat{h}_{\rho\sigma}\overline{T}^{\rho\sigma}\eta^{\mu\nu} + \widehat{h}_{\rho\sigma}\overline{C}^{\rho\sigma\mu\nu} + 2\widehat{\phi}\left(\overline{T}^{\mu\nu} - \frac{1}{4}\overline{T}\eta^{\mu\nu} + \overline{C}^{\mu\nu}\right)$$
$$+ \frac{1}{(q-1)!}\widehat{B}^\mu{}_{\rho_2\ldots\rho_q}\overline{J}^{\nu\rho_2\ldots\rho_q} - \frac{1}{2q!}\widehat{B}_{\rho_1\ldots\rho_q}\overline{J}^{\rho_1\ldots\rho_q}\eta^{\mu\nu} . \tag{74}$$

The force balance equation (56) can thereby be rewritten as

$$\overline{\nabla}_\nu \widetilde{T}^{\mu\nu} = \widetilde{f}^\mu , \tag{75}$$

in which the force density terms on the right consist just of well behaved locally source-free contributions from any incident radiation, as given by the sum

$$\widetilde{f}^\mu = \widetilde{f}_{[q]}{}^\mu + \widetilde{f}_{\mathrm{G}}{}^\mu + \widetilde{f}_{\mathrm{D}}{}^\mu , \tag{76}$$

in which each of the terms is entirely regular.

VIII.   ACTION RENORMALISATION

It has just been demonstrated that the dominant contributions to the q-form, gravitational and dilatonic self-interactions, can be described in terms a renormalised energy-momentum tensor. We now show that this renormalised energy-momentum tensor can be derived by variational methods from a renormalized action, in which the original Lagrangian master function, Λ, is replaced by an renormalised function, $\widetilde{\Lambda}$.

In order to incorporate the effects of self-interaction, as described by the renormalised force balance equation (76), it can be verified that all one needs to do is to replace the Lagrangian $\mathcal{L}$ by

$$\widetilde{\mathcal{L}} = \widehat{\Lambda} + \frac{1}{2(q!)}\widetilde{B}_{\mu_1\ldots\mu_q}\overline{J}^{\mu_1\ldots\mu_q} + \frac{1}{4}\widetilde{h}_{\mu\nu}\overline{T}^{\mu\nu} + \frac{1}{2}\widetilde{\phi}\overline{T} , \tag{77}$$

where the last three terms are the effects from non-local contributions to the fields originating far away from the brane, and the renormalised master function is $\widetilde{\Lambda} = \widehat{\Lambda} + \widehat{\Lambda}_{[q]} + \widehat{\Lambda}_{\mathrm{G}} + \widehat{\Lambda}_{\mathrm{D}}$, with renormalization terms added for each of the fields. The q-form contribution will be given by the expression

$$\widehat{\Lambda}_{[q]} = \frac{1}{2(q!)}\widehat{B}_{\mu_1\ldots\mu_q}\overline{J}^{\mu_1\ldots\mu_q} = \frac{1}{2(q!)(n-2)}\mathrm{F}_{\{\Delta,\epsilon\}} M_{[q]}^{2q+2-n}\overline{J}_{\mu_1\ldots\mu_q}\overline{J}^{\mu_1\ldots\mu_q} . \tag{78}$$

The corresponding gravitational contribution has been evaluated in [7] and has the form

$$\widehat{\Lambda}_{\mathrm{G}} = \frac{1}{4}\widehat{h}_{\mu\nu}\overline{T}^{\mu\nu} = \frac{1}{2}\mathrm{F}_{\{\Delta,\epsilon\}} M_G^{2-n}\left(\overline{T}_{\mu\nu}\overline{T}^{\mu\nu} - \frac{1}{n-2}\overline{T}^2\right) . \tag{79}$$

Finally the dilatonic contribution is obtained as

$$\widehat{\Lambda}_{\mathrm{D}} = \frac{1}{2}\widehat{\phi}\overline{T} = \frac{1}{2(n-2)}\mathrm{F}_{\{\Delta,\epsilon\}} M_D^{2-n}\overline{T}^2 . \tag{80}$$

We should point out that in performing this calculation we have included an extra factor $1/2$ which takes into account the double counting implicit in the action renormalization procedure.

Using the defining relations (33,34) and the conditions (44), it can be checked directly that the preceding prescriptions (78,79,80) actually do give rise to surface energy-momentum contributions of the forms given respectively by (69,71,73). The renormalization of the action is thus

$$\Delta\widehat{\Lambda} = \frac{1}{4}\widehat{h}_{\mu\nu}\overline{T}_{\mu\nu} + \frac{1}{2}\widehat{\phi}\overline{T} + \sum_q \frac{1}{2q!}\widehat{B}_{\mu_1\ldots\mu_q}\overline{J}^{\mu_1\ldots\mu_q}$$
$$= \frac{\mathrm{F}_{\{\Delta,\epsilon\}}}{2(n-2)}\left(M_G^{2-n}\left[(n-2)\overline{T}^{\mu\nu}\overline{T}_{\mu\nu} - \overline{T}^2\right] + M_D^{2-n}\overline{T}^2 + \sum_q \frac{M_{[q]}^{2q+2-n}}{q!}\overline{J}_{\mu_1\ldots\mu_q}\overline{J}^{\mu_1\ldots\mu_q}\right) . \tag{81}$$

The validity of this coherence criterion means that the "dressed" brane model characterised by the action density, $\widetilde{\Lambda}$, will indeed have a corresponding "dressed" surface stress momentum energy density tensor $\widetilde{T}^{\mu\nu}$ of the required form, as given by (74).

IX. MAXIMAL $(p+1)$-FORM AND NAMBU-GOTO BRANE

An important special case, including that of an ordinary axionic coupling to a string in 4-dimensional spacetime [15], is that for which the number of indices of the form field $B_{\mu_1\ldots\mu_q}$ is equal to the dimension of the brane worldsheet. In this case, $q = p+1$, the corresponding source tensor on the brane must simply be proportional to the corresponding surface measure tensor, that is,

$$\overline{J}^{\mu_0\cdots\mu_p} = \bar{\kappa}\,\mathcal{E}^{\mu_0\cdots\mu_p} \tag{82}$$

with a proportionality coefficient $\bar{\kappa}$ having a uniform value over the worldsheet in order for the conservation condition (22) to be satisfied. It follows that the regularized action contribution will have the *constant* form

$$\widehat{\Lambda}_{[p+1]} = -\frac{1}{2(n-2)}\mathrm{F}_{\{\Delta,\epsilon\}}\,M_{[p+1]}^{2p+4-n}\,\bar{\kappa}^2 \,. \tag{83}$$

The manifestly negative definite nature of this axionic contribution is what makes it possible in particular cases [15] for it to be cancelled by the corresponding, manifestly positive definite, contribution (80) from the dilaton, as discussed below.

It is worthwhile to consider the simplest dimensionally unrestricted application, which is to a NG p-brane, that is, one for which the master function Λ is just a constant, which we can express in terms of a mass scale, as

$$\overline{\Lambda} = -M_K^{\mathrm{p}+1}\,, \tag{84}$$

where M_K is a fixed mass scale that will be referred to as the Kibble mass to distinguish it from other mass scales in the theory. In the context of Superstring theory this quantity M_K is usually supposed to be of the order of magnitude of the Planck mass M_{Pl}, whereas in the context of cosmic string theory it is generally expected that it should be of the same order of magnitude as the Higgs expectation value related to symmetry breaking.

In this special case, the energy-momentum tensor is of course simply proportional to the fundamental tensor,

$$\overline{T}^{\mu\nu} = -M_K^{\mathrm{p}+1}\eta^{\mu\nu}\,, \tag{85}$$

so its trace will be given by

$$\overline{T} = -(p+1)M_K^{\mathrm{p}+1}\,. \tag{86}$$

The corresponding the hyper-Cauchy tensor is found [4] to be

$$\overline{\mathcal{C}}^{\mu\nu\rho\sigma} = M_K^{\mathrm{p}+1}\big(\eta^{\mu(\rho}\eta^{\sigma)\nu} - \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\big) \quad\Longrightarrow\quad \overline{\mathcal{C}}^{\mu\nu} = \frac{1-p}{2}M_K^{\mathrm{p}+1}\eta^{\mu\nu}\,, \tag{87}$$

so it is apparent that $\overline{\mathcal{C}}^{\mu\nu}$ will vanish in the string case, $p = 1$. The combination involved in the expression (60) for the dilatonic force density will be given by

$$\overline{T}^{\mu\nu} + \overline{\mathcal{C}}^{\mu\nu} = -\frac{p+1}{2}M_K^{\mathrm{p}+1}\eta^{\mu\nu}\,, \tag{88}$$

which never vanishes.

The dynamical equation of motion (56) can be seen from (7) to reduce to the form

$$M_K^{\mathrm{p}+1}K^\rho = -f^\rho\,, \tag{89}$$

in which, by (59), the gravitational contribution to the surface force on the right hand side can be written [4] in the form

$$\widehat{f}_{\mathrm{G}}{}^\mu = M_K^{\mathrm{p}+1}\left(\perp^{\mu\nu}\eta^{\rho\sigma}\big(\nabla_\rho h_{\nu\sigma} - \frac{1}{2}\nabla_\nu h_{\rho\sigma}\big) + \big(\perp^{\mu\nu}K^\rho + \frac{1}{2}\eta^{\rho\nu}K^\mu - K^{\nu\rho\mu}\big)h_{\nu\rho}\right)\,, \tag{90}$$

while the corresponding dilatonic contribution will be given by

$$\widehat{f}_{\mathrm{D}}{}^\mu = (p+1)\,M_K^{\mathrm{p}+1}\left(\phi\,K^\mu - \perp^{\mu\nu}\nabla_\nu\phi\right)\,. \tag{91}$$

Much of the early work on perturbations of cosmic strings is flawed by the omission of both the gradient terms and the orthogonal projection operator, $\perp^{\mu\nu}$, in (90). One of the reasons the problem was not noticed in earlier studies

of NG string self-interactions is that, in the particular case when $h_{\mu\nu}$ is due just to self interaction, the dominant short-range contribution responsible for the divergence turns out to be restricted in such a way as to give a result that does satisfy the orthogonality requirement. However, other forces, such as those due to external sources would require the correct formulae (90,91) to be used.

We can write out the full expression for the self-force acting on the brane as

$$\widehat{f}^{\mu} = \left[(p+1)(n-p-3)M_K^{2p+2}M_G^{2-n} + (p+1)^2 M_K^{2p+2}M_D^{2-n} - \bar{\kappa}^2 M_{[p+1]}^{2p+4-n} \right] \frac{\mathrm{F}_{\{\Delta,\epsilon\}}}{2(n-2)} K^{\mu} , \tag{92}$$

and, the renormalization of the action as

$$\Delta\widehat{\Lambda} = \left[(p+1)(n-p-3)M_K^{2p+2}M_G^{2-n} + (p+1)^2 M_K^{2p+2}M_D^{2-n} - \bar{\kappa}^2 M_{[p+1]}^{2p+4-n} \right] \frac{\mathrm{F}_{\{\Delta,\epsilon\}}}{2(n-2)} . \tag{93}$$

A. Previously derived special cases

A number of special cases were discussed in section I. The first was the point particle in four dimensions coupled to electromagnetism, that is, $p = 0$, $n = 4$, $M_G = M_D = 0$ in which case

$$\widehat{f}^{\mu} = -\frac{\bar{\kappa}^2}{4}\mathrm{F}_{\{\Delta,\epsilon\}}K^{\mu} , \qquad \Delta\widehat{\Lambda} = -\frac{\bar{\kappa}^2}{4}\mathrm{F}_{\{\Delta,\epsilon\}} , \tag{94}$$

where $\bar{\kappa} = e$ the electromagnetic coupling, $\mathrm{F}_{\{\Delta,\epsilon\}} = (\pi\epsilon)^{-1}$ and ϵ corresponds to the radius of the electron. Another is the case of the global, or axion, string which corresponds to $p = 1$, $n = 4$, $M_G = M_D = 0$ and hence

$$\widehat{f}^{\mu} = -\frac{\bar{\kappa}^2}{4}M_{[2]}^2 K^{\mu}\mathrm{F}_{\{\Delta,\epsilon\}} , \qquad \Delta\widehat{\Lambda} = -\frac{\bar{\kappa}^2}{4}M_{[2]}^2\mathrm{F}_{\{\Delta,\epsilon\}} , \tag{95}$$

with $\bar{\kappa} = 1$, $M_{[2]} = \sqrt{2\pi}f_a$ and $\mathrm{F}_{\{\Delta,\epsilon\}} = 4\log(\Delta/\epsilon)$.

The case of pure gravity ($M_D = M_{[p+1]} = 0$) was considered in [17] and for this case the calculation presented here yields

$$\widehat{f}^{\mu} = (p+1)(n-p-3)M_K^{2p+2}M_G^{2-n}\mathrm{F}_{\{\Delta,\epsilon\}} , \qquad \Delta\widehat{\Lambda} = (p+1)(n-p-3)M_K^{2p+2}M_G^{2-n}\mathrm{F}_{\{\Delta,\epsilon\}} , \tag{96}$$

where, in the notation of [17], $\lambda = M_K^{p+1}$ and $G = M_G^{2-n}$.

B. Co-dimension two

The case of co-dimension two ($n = p+3$) is of interest since the gravitational self-force is exactly zero. The dilatonic and form-fields have opposite signs and their respective self-forces can be made to cancel if

$$\frac{M_K^2}{M_D M_{[p+1]}} = \left(\frac{\bar{\kappa}}{p+1} \right)^{\frac{2}{p+1}} . \tag{97}$$

For the special case of $n = 4$, $p = 1$, this corresponds to the case previously discussed [12, 13, 14, 15] where

$$M_K^2 = \frac{1}{2}\bar{\kappa}M_D M_{[2]} . \tag{98}$$

In the notation of [14] $\alpha = M_G/M_D$, $\lambda = \bar{\kappa}M_G M_{[2]}/2$ and $\mu = M_K^2$, and hence the condition (98) requires that $\alpha = 1$ and $\lambda = \mu$.

The co-dimension two case is also interesting when $n = 6$ in the context of brane-world models. Many authors have observed that, for a four-dimensional brane-world in six dimensions, the gravitational effect of the bare tension of the brane is to produce a conical deficit in the spacetime which is unobservable to a brane-based observer. Most of the work in the literature has considered specific solutions and infinitely thin branes; our work has extended this to allow for very general brane configurations which are extrinsically curved within the spacetime [17] and can account for branes of finite thickness [16]. This current paper includes the effect of the dilaton and form-field effects which

can cancel each other with an appropriate choice of the mass scales M_D, M_K and $M_{[4]}$; this choice of coupling is the one selected naturally by certain sting models, as described in [12].

The fact that the self-force is zero in the co-dimension two case has obvious implications for the cosmological constant problem. If a vacuum energy component of matter on the brane does not gravitate, the excessive values predicted by quantum field theories no longer present a fine-tuning problem. This is not a complete solution, however, because the bulk could still induce a cosmological acceleration on the brane. Furthermore, cosmological inflation would not be able to operate by the usual mechanism in such models.

X.   DISCUSSION AND CONCLUSIONS

We have calculated the self-force of branes due to classical gravitational, dilatonic and form-field mediated interactions. This analysis is very general, accounting for finite thickness effects via the analysis in [16] and allowing for the brane to be curved provided that extrinsic curvature scale is long compared to the ultra-violet cut-off associated with the brane thickness. We have also expressed these self-interactions as renormalizations of the action.

For the gravitational force alone, the force is proportional to the extrinsic curvature vector K^μ. In the special case of codimension two, the force is actually zero, so the self-interactions cancel. This is a well-known result for cosmic strings in four dimensions; the tension of the string determines the conical deficit angle but does not affect the geometry away from the string or the induced metric on the string. More recently, this is has been fashionable in the context of co-dimension two brane-worlds as a possible resolution of the cosmological constant problem.

When a dilaton and a maximal q-form are included, certain combinations of couplings result in zero total self-force. Many Superstring theories select precisely those couplings, suggesting that our calculation could be used as a consistency relation for models, i.e., that the self-force should always be zero for extended objects in a consistent fundamental theory.

The formalism could be applied to many situations where there is an extended object in a spacetime. In particular it could be applied to the Dirac–Born–Infeld (DBI) action which is of interest in string theory when studying D-branes.

Acknowledgments

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[29] In line with the usual convention, we use round/square brackets to denote index symmetrization/antisymmetrization